Regularization of Gauge Theory on Noncommutative $\mathbb{R}^4$

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Outline

- Noncommutativity
- Gauge Theory on $\mathbb{R}^4_\theta$
- Regularization of the Gauge Theory
- Instantons
- Conclusions

Noncommutativity

- Infinities in QFT suggest new structures beyond
- Noncommutativity

$$[x_i, x_j] = i\theta_{ij}$$

  as a model of quantized spacetime
- In the canonical case $\theta_{ij} \in \mathbb{R}$ singularities are not cured
- Regularization of the noncommutative theory necessary

Canonical Noncommutative $\mathbb{R}^4$

- In the Euclidian, the coordinates can be made to form two Heisenberg algebras

$$[x_L^+, x_L^-] = \theta, \;\; [x_R^+, x_R^-] = \theta, \;\; [x_L^\pm, x_R^\pm] = 0$$

- Representation as creator and annihilator operators on usual Fock space

- Derivatives are inner: $\partial_i \;\hat{=}\; -\frac{i}{\theta}[x_i, \,\cdot\,]$

Gauge Theory

- Matrix action with ground state $X_i = x_i$

$$S_\theta = -\frac{(2\pi)^2}{2g^2\theta^2} tr\left([X_i, X_j] - i\theta_{ij}\right)^2$$

- Covariant coordinates $X_i = x_i + A_i$

$$\begin{aligned} X_i &\rightarrow UX_iU^\dagger \\ A_i &\rightarrow U[x_i, U^\dagger] + UA_iU^\dagger \end{aligned}$$

- Field strength

$$iF_{ij} = [X_i, X_j] - i\theta_{ij} = [x_i, A_j] - [x_j, A_i] + [A_i, A_j]$$

Problems

- Rank of the gauge group not fixed
- Infinities of the commutative theory persist
- New infinities: IV/UR-mixing

$\longrightarrow$ **Regularization** necessary

The Fuzzy Sphere

For the regularization, we need a finite-dimensional representation, that in some limit goes to $\mathbb{R}^4_\theta$. The fuzzy sphere S^2_M is used as a building block:

- creators λ_i with $i \in \{1, 2, 3\}$ and relations
$$[\lambda_i, \lambda_j] = i\epsilon_{ijk}\lambda_k \quad \text{and} \quad \sum_i \lambda_i\lambda_i = \frac{M^2 - 1}{4}$$
- M-dim. Irreps of $su(2)$
- Anaolgs of spherical harmonics up to an angular momentum $M - 1$.
- Coordinates x_i correspond to $x_i \approx \frac{2R}{M}\lambda_i$, tangential derivatives $J_i = [\lambda_i, \cdot\,]$

A four-dimensional space

- To go to four dimensions, we simply take two copies of the fuzzy sphere
- This new space $S^2_M \times S^2_M$ is generated by the M^2-dimensional matrices

$$\lambda_{iL} = \lambda_i \otimes 1 \quad \text{and} \quad \lambda_{iR} = 1 \otimes \lambda_i$$

- Coordinates scale again as $x_{iL/R} \approx \frac{2R}{M}\lambda_{iL/R}$, the tangential derivatives as $J_{iL/R} = [\lambda_{iL/R}, \cdot\,]$

Gauge Theory on $S^2_M \times S^2_M$

To do gauge theory, we use an action over six M^2-dimensional matrices $B_\mu = (B_{iL}, B_{iR})$

$$S_{fuzzy} = \frac{8\pi^2}{M^2}\mathrm{tr}((i[B_{iL/R}, B_{jL/R}] + \epsilon_{ijk} B_{kL/R})^2 - [B_{iL}, B_{jR}]^2 + V(B))$$

- The ground state obviously is $\lambda_{iL/R}$
- The covariant coordinates $B_{iL/R} = \lambda_{iL/R} + A_{iL/R}$ form the gauge theory
- The potential $V(B) = 2(B_{iL}B_{iL} - \frac{M^2-1}{4})^2 + 2(B_{iR}B_{iR} - \frac{M^2-1}{4})^2$ stabilizes the size of the representation

Finiteness of the model

- The gauge group is fixed to be $U(1)$
- Gauge groups $U(N)$ are implemented by changing the size of the matrices to NM^2
- Quantization can be performed via a path integral over the matrix entries

$$Z[J] = \int dB_\mu e^{-S[B_\mu]+tr\, B_\mu J_\mu}.$$

It is well defined and finite for every M

Scaling limit to $\mathbb{R}^4_\theta$

- The north pole (i.e. $\lambda_3 \approx \frac{M}{2}$) of S^2_M is blown up by a double scaling limit $R, M \to \infty$, keeping $\frac{2R^2}{M} = \theta$ fixed

$$[x_1, x_2] \;=\; i\frac{2R}{M}\sqrt{R^2 - x_1^2 - x_2^2} \;=\; i\theta + O(1/M).$$

  This is nothing but $\mathbb{R}^2_\theta$

- We get the coordinates x_i with $i \in \{1, ..., 4\}$ of $\mathbb{R}^4_\theta$ by scaling λ_{iL} and λ_{iR} as

$$\sqrt{\frac{2\theta}{M}}\lambda_{1,2\,L} \to x_{1,2} \quad \text{and} \quad \sqrt{\frac{2\theta}{M}}\lambda_{1,2\,R} \to x_{3,4}$$

Scaling limit of the gauge theory

- Under the double scaling limit, the covariant coordinates B_μ on $S^2_M \times S^2_M$ are mapped to the covariant coordinates X_μ on $\mathbb{R}^4_\theta$

$$\sqrt{\frac{2\theta}{M}} B_{1,2/,L} \to X_{1,2} \quad \text{and} \quad \sqrt{\frac{2\theta}{M}} B_{1,2/,R} \to X_{3,4}$$

- The complete gauge theory from $S^2_M \times S^2_M$ is mapped to $\mathbb{R}^4_\theta$, we get

$$S_{fuzzy} \to S_\theta$$

- This is the desired **regularization** for gauge theory on $\mathbb{R}^4_\theta$

Instantons

- On $\mathbb{R}^4_\theta$, a class of instantons can be constructed by setting

$$X_i = \begin{pmatrix} diag(c_{1,i}, ..., c_{k,i}) & 0 \\ 0 & x_i \end{pmatrix},$$

fulfilling the EOMs $[X_i, F_{ij}] = 0$

- On $S^2_M \times S^2_M$, we can mimic this by setting

$$\begin{aligned} B_{iL} &= \alpha \begin{pmatrix} diag(d_{1,Li}, ..., d_{k,Li}) & 0 \\ 0 & \lambda'_i \otimes 1_{M'' \times M''} \end{pmatrix}, \\ B_{iR} &= \alpha \begin{pmatrix} diag(d_{1,Ri}, ..., d_{k,Ri}) & 0 \\ 0 & 1_{M' \times M'} \otimes \lambda''_i \end{pmatrix}, \end{aligned}$$

fulfilling the EOMs on $S^2_M \times S^2_M$

Instantons

- On $\mathbb{R}^4_\theta$, we could have any charge k of the instantons
- On $S^2_M \times S^2_M$, the overall size of the the matrix B_μ is fixed to M^2. For finite instantons, we need

$$M' = M - l, \; M'' = M + l \; \text{ and } \; k = l^2$$

- We get back the instantons on $\mathbb{R}^4_\theta$, but the regularization produces a superselection rule for the instanton charge

Conclusions

- Gauge theory on $\mathbb{R}^4_\theta$ must be regularized (because of infinities and sectors for every rank of the gauge group) to be well defined
- Gauge theory on $S^2_M \times S^2_M$ provides such a regularization
- We were able to match parts of the instanton sectors on $S^2_M \times S^2_M$ and $\mathbb{R}^4_\theta$
- The regularization introduces a superselection rule for the instanton charge