Billiards and pencils of quadrics

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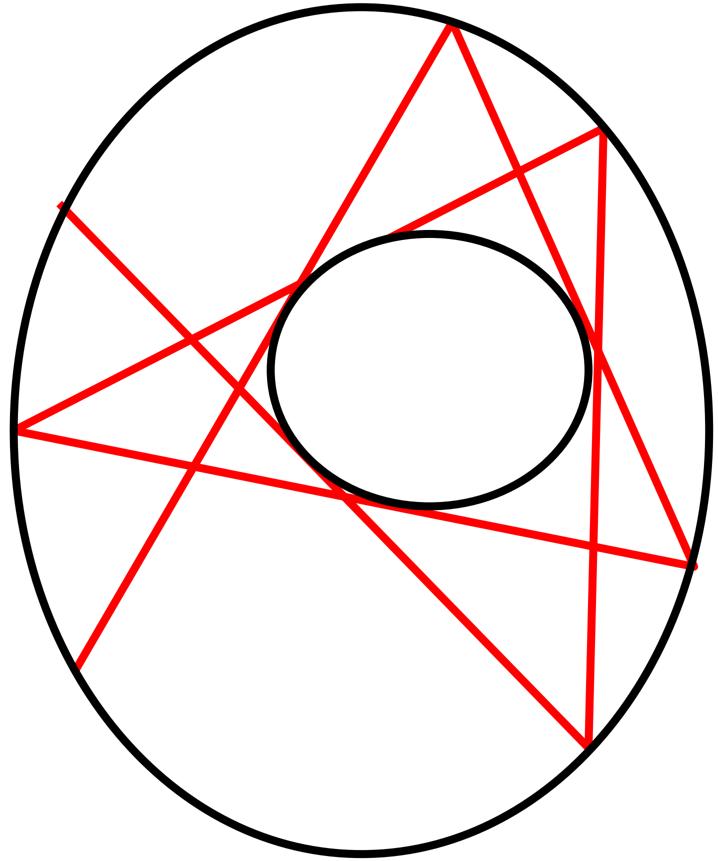
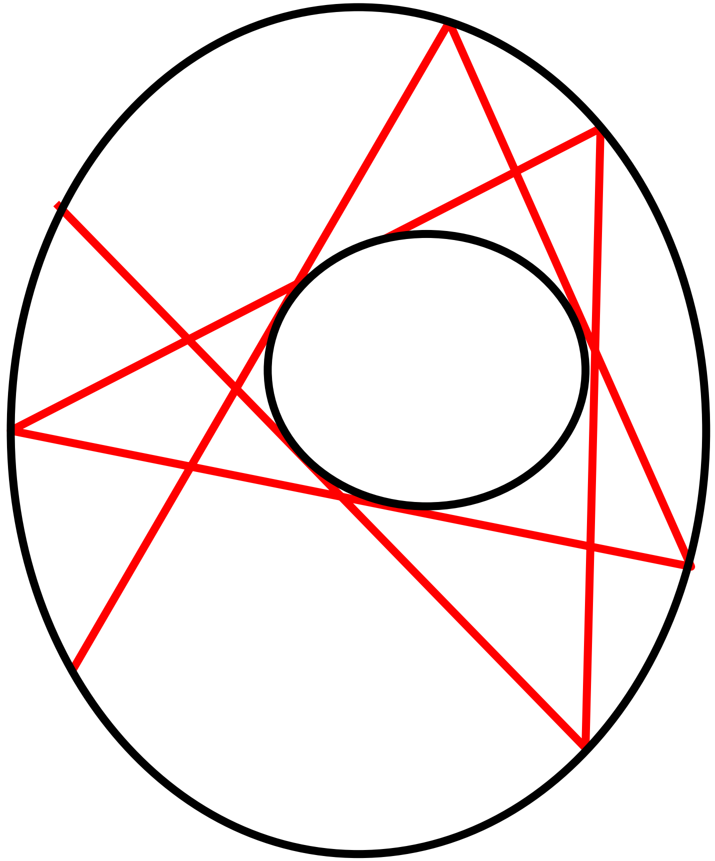
Poncelet theorem

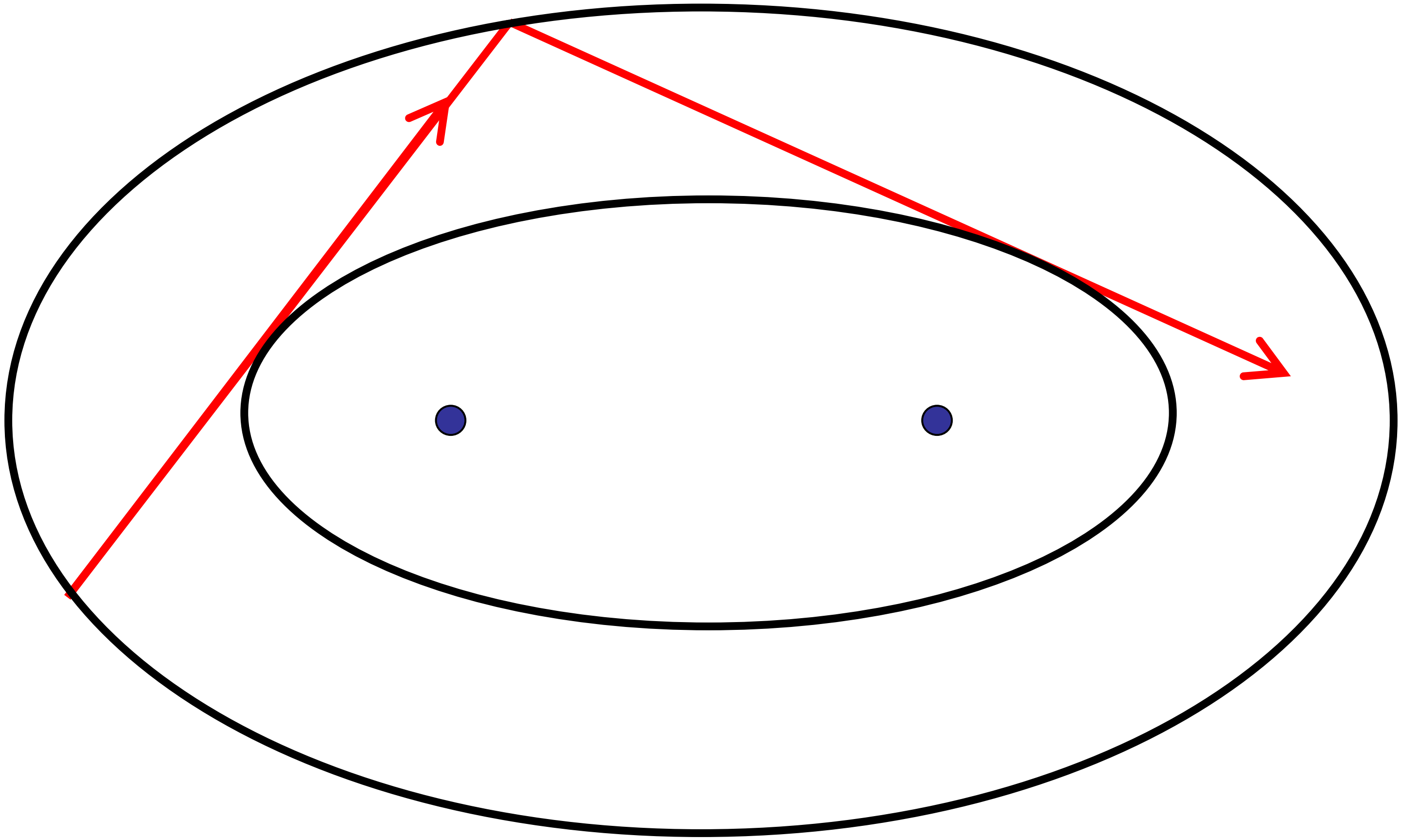


Poncelet theorem

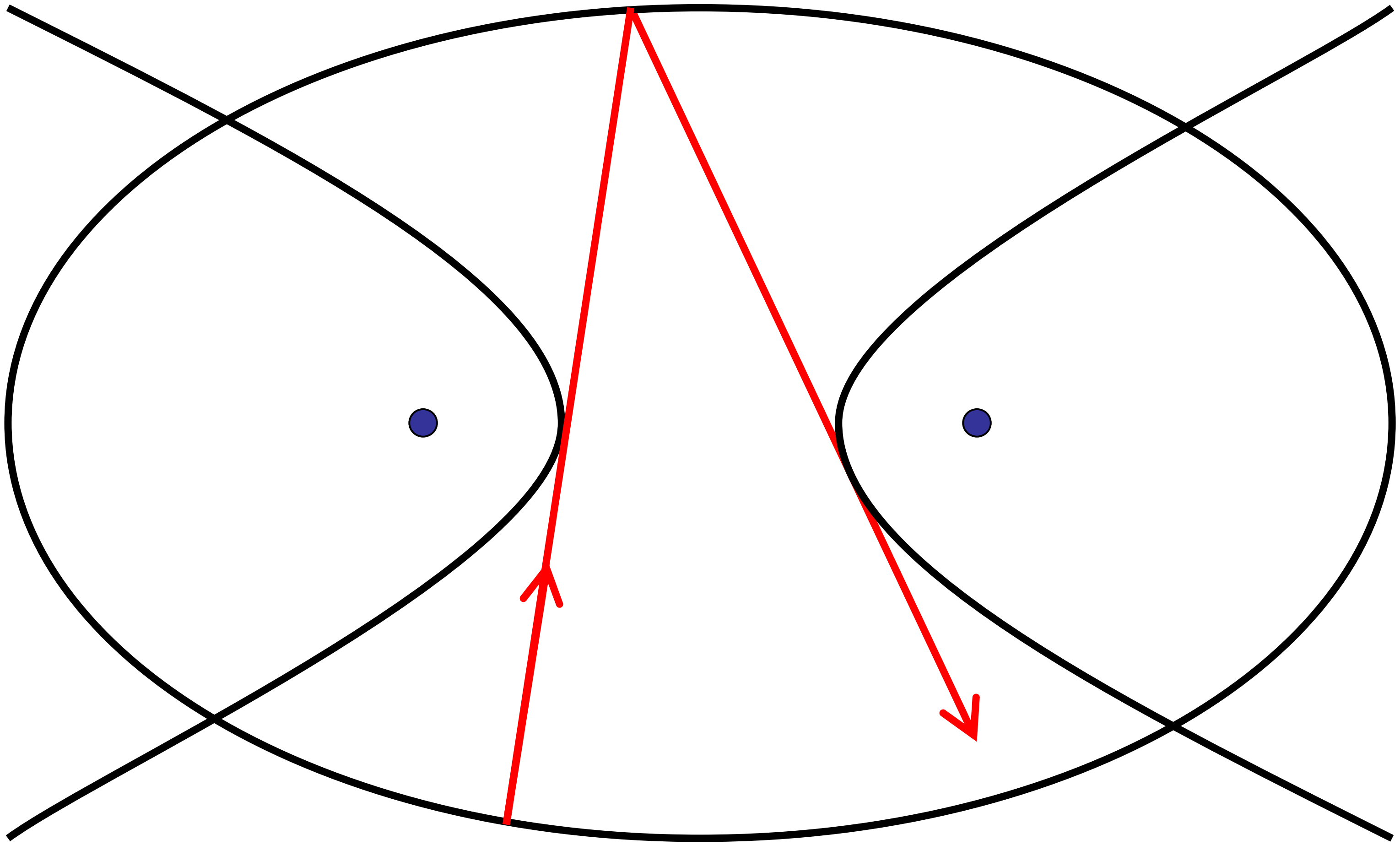
Suppose that two ellipses are given in the plane, together with a closed polygonal line inscribed in one of them and circumscribed about the other one. Then, Poncelet theorem states that infinitely many such closed polygonal lines exist - every point of the first ellipse is a vertex of such a polygon. Besides, all these polygons have the same number of sides.



Mechanical interpretation of Poncelet theorem



Mechanical interpretation of Poncelet theorem



Classical Cayley's condition

Let *F* be a pencil given by conics *C* and *G*.

There exists a polygon with *n* sides inscribed in *G* and circumscribed about *C* if and only if

$$\begin{vmatrix} C_3 & C_4 & \dots & C_{p+1} \\ C_4 & C_5 & \dots & C_{p+2} \\ & & \dots & \\ C_{p+1} & C_{p+2} & \dots & C_{2p-1} \end{vmatrix} = 0$$

for $n=2p$

$$\begin{vmatrix} C_2 & C_3 & \dots & C_{p+1} \\ C_3 & C_4 & \dots & C_{p+2} \\ & & \dots & \\ C_{p+1} & C_{p+2} & \dots & C_{2p} \end{vmatrix} = 0$$

for $n=2p+1$

$$\sqrt{D(x)} = A + Bx + C_2x^2 + C_3x^3 + \dots .$$

$D(x)$ is the discriminant of the conic $C+xG$.

Cayley-type condition for billiard inside ellipsoid in arbitrary dimensional space is derived in:

V. Dragović, M. Radnović, *Conditions of Cayley's type for ellipsoidal billiard*, J. Math. Phys. **39** (1998), no. 1, 355-362.

V. Dragović, M. Radnović, *On periodical trajectories of the billiard systems within an ellipsoid in* $\mathbb{R}^d$ *and generalized Cayley's condition*, J. Math. Phys. **39** (1998), no. 11, 5866-5869.

Recent result:

Cayley-type condition for periodical billiard trajectories inside *k* confocal quadrics in *d*-dimensional space.

V. Dragović, M. Radnović, *Cayley-type conditions for billiards within k quadrics in* $\mathbb{R}^d$, J. of Phys. A: Math. Gen. **37** (2004), 1269-1276.

Billiard inside a domain bounded by confocal quadrics

Theorem *A trajectory of the billiard system within* Ω *with caustics* $\mathcal{Q}_{\alpha_1}, \dots,$ $\mathcal{Q}_{\alpha_{d-1}}$ *is periodic with exactly* n_s *points at* $\mathcal{Q}_{\gamma'_s}$ *and* n_s *points at* $\mathcal{Q}_{\gamma''_s}$ $(1 \le s \le d)$ *if and only if*

$$\sum_{s=1}^{d} n_s \left(\mathcal{A}(P_{\gamma'_s}) - \mathcal{A}(P_{\gamma''_s})\right) = 0$$

on the Jacobian of the curve

$$\Gamma \;:\; y^2 = \mathcal{P}(x) := (a_1 - x)\cdots(a_d - x)(\alpha_1 - x)\cdots(\alpha_{d-1} - x).$$

Here, $\mathcal{A}$ *denotes the Abel-Jacobi map, where* $P_{\gamma'_s}$, $P_{\gamma''_s}$ *are points on* Γ *with coordinates* $P_{\gamma'_s} = \left(\gamma'_s, (-1)^s\sqrt{\mathcal{P}(\gamma'_s)}\right)$, $P_{\gamma''_s} = \left(\gamma''_s, (-1)^s\sqrt{\mathcal{P}(\gamma''_s)}\right)$.

Example Consider two domains Ω' and Ω'' in $\mathbb{R}^3$. Let Ω' be bounded by the ellipsoid $\mathcal{Q}_0$ and the two-folded hyperboloid $\mathcal{Q}_\beta$, $a_2 < \beta < a_1$, in such a way that Ω' is placed between the branches of $\mathcal{Q}_\beta$. On the other hand, suppose Ω'' is bounded by $\mathcal{Q}_0$, the righthand branch of $\mathcal{Q}_\beta$ (this one which is placed in the half-space $x_1 > 0$) and the plane $x_3 = 0$. Elliptic coordinates of points inside both Ω' and Ω'' satisfy:

$$0 \le \lambda_3 \le a_3, \quad \beta \le \lambda_1 \le a_1.$$

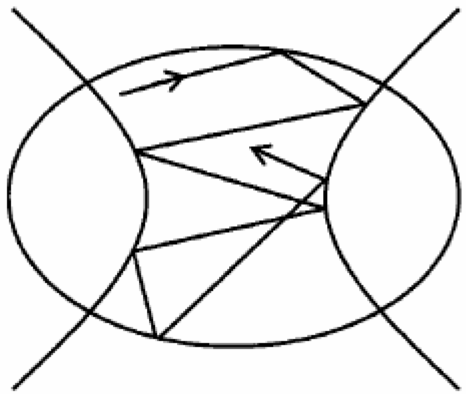
Consider billiard trajectories within these two domains, with caustics $\mathcal{Q}_{\mu_1}$ and $\mathcal{Q}_{\mu_2}$, $a_3 < \mu_1 < a_2$, $a_2 < \mu_2 < a_1$.

In Ω'', existence of a periodic trajectory with caustics $\mathcal{Q}_{\mu_1}$ and $\mathcal{Q}_{\mu_2}$, which becomes closed after n bounces at $\mathcal{Q}_0$ and $2m$ bounces at $\mathcal{Q}_\beta$ is equivalent to the equality:

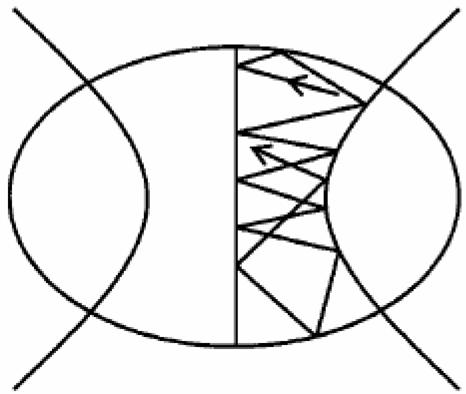
$$n\big(\mathcal{A}(P_0) - \mathcal{A}(P_{a_3})\big) + 2m\big(\mathcal{A}(P_\beta) - \mathcal{A}(P_{\mu_1})\big) = 0,$$

on the Jacobian of the corresponding hyperelliptic curve. In Ω', existence of a trajectory with same properties is equivalent to:

$$n\big(\mathcal{A}(P_0) - \mathcal{A}(P_{a_3})\big) + 2m\big(\mathcal{A}(P_\beta) - \mathcal{A}(P_{\mu_1})\big) = 0 \quad \text{and} \quad n \quad \text{is even.}$$



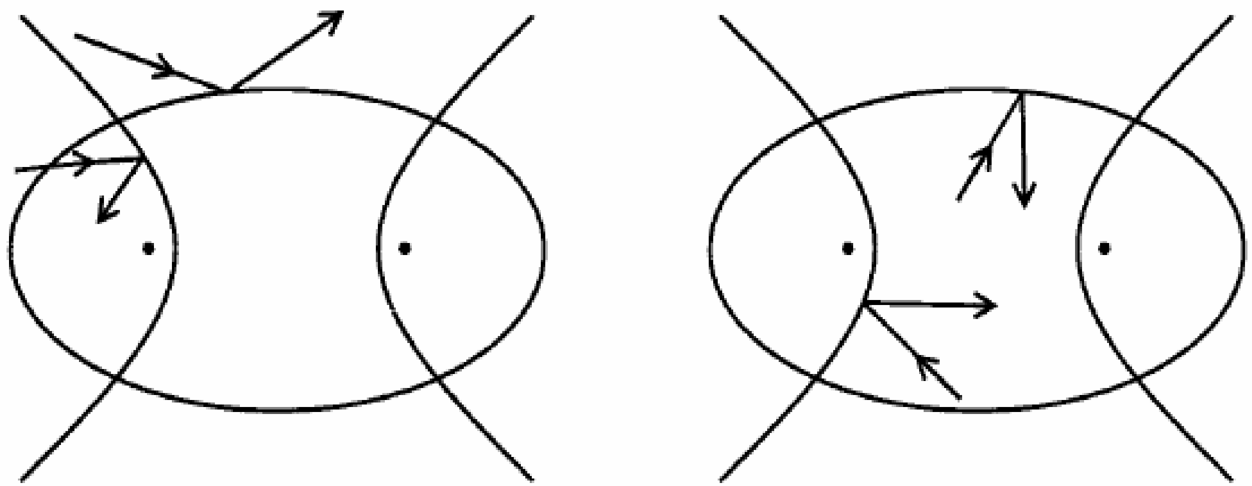
A trajectory inside Ω'



A trajectory inside Ω''

Billiard ordered game

Definition A ray reflects *from outside* at the quadric Q_λ if the reflection point is a local maximum of the Jacobian coordinate λ_s, and it reflects *from inside* if the reflection point is a local minimum of the coordinate λ_s.



Reflection from outside   Reflection from inside

Definition *The billiard ordered game* joined to quadrics $\mathcal{Q}_{\beta_1}, \dots, \mathcal{Q}_{\beta_k}$, with the *signature* $(i_1, \dots, i_k)$ is the billiard system with trajectories having bounces at $\mathcal{Q}_{\beta_1}, \dots, \mathcal{Q}_{\beta_k}$ respectively, such that

$$\text{the reflection at } \mathcal{Q}_{\beta_s} \text{ is from inside if } i_s = +1;$$
$$\text{the reflection at } \mathcal{Q}_{\beta_s} \text{ is from outside if } i_s = -1.$$

Note that any trajectory of a billiard ordered game has $d-1$ caustics from the same family

Suppose $\mathcal{Q}_{\beta_1}, \dots, \mathcal{Q}_{\beta_k}$ are ellipsoids and consider a billiard ordered game with the signature $(i_1, \dots, i_k)$. In order that trajectories of such a game stay bounded, the following condition has to be satisfied:

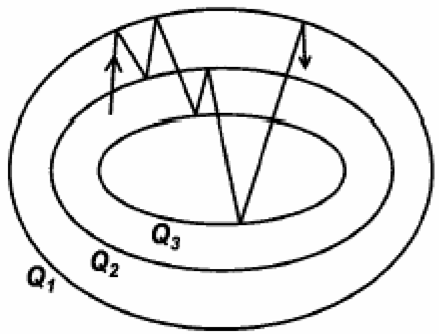
$$i_s = -1 \;\Rightarrow\; i_{s+1} = i_{s-1} = 1 \text{ and } \beta_{s+1} < \beta_s,\ \beta_{s-1} < \beta_s.$$

(Here, we identify indices 0 and $k+1$ with k and 1 respectively.)

Example On Figure a trajectory corresponding to the 7-tuple

$$(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_1, \mathcal{Q}_3, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_1),$$

with the signature $(1, -1, 1, -1, 1, 1, 1)$, is shown.

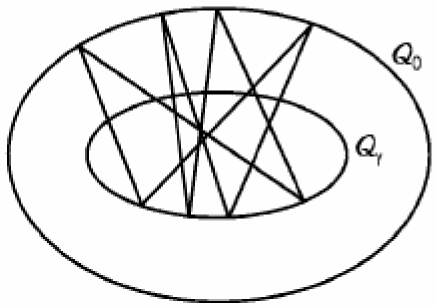


Theorem 5 *Given a billiard ordered game within k ellipsoids $\mathcal{Q}_{\beta_1}, \dots, \mathcal{Q}_{\beta_k}$ with the signature $(i_1, \dots, i_k)$. Its trajectory with caustics $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-1}}$ is k-periodic if and only if*

$$\sum_{s=1}^{k} i_s\big(\mathcal{A}(P_{\beta_s}) - \mathcal{A}(P_\alpha)\big)$$

is equal to a sum of several expressions of the form: $\Big(\mathcal{A}(P_{\alpha_p}) - \mathcal{A}(P_{\alpha_{p'}})\Big)$ on the Jacobian of the curve $\Gamma : y^2 = \mathcal{P}(x)$, where $P_{\beta_s} = \Big(\beta_s, +\sqrt{\mathcal{P}(\beta_s)}\Big)$, $\alpha = \min\{a_d, \alpha_1, \dots, \alpha_{d-1}\}$ and $\mathcal{Q}_{\alpha_p}$, $\mathcal{Q}_{\alpha_{p'}}$ are pairs of caustics of the same type.

When $\mathcal{Q}_{\beta_1} = \cdots = \mathcal{Q}_{\beta_k}$ and $i_1 = \cdots = i_k = 1$ we obtain the Cayley-type condition for the billiard motion inside an ellipsoid in $\mathbb{R}^d$.



A closed trajectory of the billiard ordered game with 8 alternate bounces from inside of two ellipses

The explicit condition for periodicity of such a trajectory is:

$$\mathrm{rank} X < 2,$$

with

$$\begin{aligned}
X_{11} &= -4C_0 + C_1\gamma + 3B_1\gamma + 2B_2\gamma^2 + B_3\gamma^3 \\
X_{12} &= -3C_0 + C_1\gamma + 3B_0 + 2B_1\gamma + B_2\gamma^2 \\
X_{21} &= -6C_0 + C_2\gamma^2 + 6B_0 + 6B_1\gamma + 5B_2\gamma^2 + 3B_3\gamma^3 \\
X_{22} &= -6C_0 + C_1\gamma + C_2\gamma^2 + 6B_0 + 5B_1\gamma + 3B_2\gamma^2 \\
X_{31} &= -4C_0 + C_3\gamma^3 + 4B_0 + 4B_1\gamma + 4B_2\gamma^2 + 3B_3\gamma^3 \\
X_{32} &= -4C_0 + C_2\gamma^2 + C_3\gamma^3 + 4B_0 + 4B_1\gamma + 3B_2\gamma^2.
\end{aligned}$$