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## **Lorentz symmetry on deformed spaces**

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# Outline

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## Deformed (noncommutative) space

Commutative space generated by coordinates  $x^\mu$ ,  
 $\mu = 1, \dots, n$  with

$$[x^\mu, x^\nu] = 0.$$

Noncommutative space  $\hat{A}_{\hat{x}}$ , generated by  $\hat{x}^\mu$  coordinates,  
 $\mu = 1, \dots, n$  such that:

$$[\hat{x}^\mu, \hat{x}^\nu] = \Theta^{\mu\nu}(\hat{x}). \quad (1)$$

It is the associative free algebra generated by  $\hat{x}^\mu$  and divided by ideal generated by relations (1).

In other words, it consists of all polynomials of  $\hat{x}^\mu$  and if one polynomial can be transformed to another one using (1) they are considered equal.

$$\begin{aligned} \hat{x}^1 \hat{x}^2 &= [\hat{x}^1, \hat{x}^2] + \hat{x}^2 \hat{x}^1 \\ &= \Theta^{12}(\hat{x}) + \hat{x}^2 \hat{x}^1 \end{aligned}$$

Three forms of  $\Theta^{\mu\nu}(\hat{x})$  of special interest

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \quad \theta^{\mu\nu} = -\theta^{\nu\mu} = \text{const.}, \quad (2)$$

$$[\hat{x}^\mu, \hat{x}^\nu] = iC_\lambda^{\mu\nu} \hat{x}^\lambda, \quad C_\lambda^{\mu\nu} \text{ Lie algebra struct. const.}, \quad (3)$$

$$\hat{x}^\mu \hat{x}^\nu = \frac{1}{q} R^{\mu\nu}_{\rho\sigma} \hat{x}^\rho \hat{x}^\sigma \quad R \text{ matrix of quantum group.} \quad (4)$$

## $\theta$ -deformed space

$$\begin{aligned} [\hat{x}^\mu, \hat{x}^\nu] &= i\theta^{\mu\nu}, \\ \theta^{\mu\nu} &= -\theta^{\nu\mu} = \text{const. and } \theta^{\mu\nu} \in \mathbb{R}. \end{aligned} \quad (5)$$

## Derivatives

Map  $\hat{\partial}$  such that  $\hat{\partial} : \hat{\mathcal{A}}_{\hat{x}} \rightarrow \hat{\mathcal{A}}_{\hat{x}}$  (6)

defined on coordinates

$$[\hat{\partial}_\rho, \hat{x}^\mu] = \delta_\rho^\mu + \text{additional terms} \quad (7)$$

it has to be consistent with (5) relations

$$\hat{\partial}_\rho \left( [\hat{x}^\mu, \hat{x}^\nu] - i\theta^{\mu\nu} \right) = 0. \quad (8)$$

Solution

$$[\hat{\partial}_\rho, \hat{x}^\mu] = \delta_\rho^\mu. \quad (9)$$

Also,

$$[\hat{\partial}_\rho, \hat{\partial}_\sigma] = 0 \quad (10)$$

is compatible with (9),

$$\hat{\partial}_\sigma \left( [\hat{\partial}_\rho, \hat{x}^\mu] - \delta_\rho^\mu \right) = \left( [\hat{\partial}_\rho, \hat{x}^\mu] - \delta_\rho^\mu \right) \hat{\partial}_\sigma = 0.$$

Leibniz rule

$$\hat{\partial}_\rho (\hat{f}\hat{g}) = (\hat{\partial}_\rho \hat{f})\hat{g} + \hat{f}(\hat{\partial}_\rho \hat{g}). \quad (11)$$

## Hopf algebra of derivatives

algebraic relations

$$[\hat{\partial}_\rho, \hat{\partial}_\sigma] = 0, \quad (12)$$

comultiplication

$$\Delta \hat{\partial}_\rho = \hat{\partial}_\rho \otimes 1 + 1 \otimes \hat{\partial}_\rho, \quad (13)$$

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta \quad \text{and} \quad [\Delta \hat{\partial}_\mu, \Delta \hat{\partial}_\nu] = 0,$$

counit and antipode

$$\varepsilon(\hat{\partial}_\rho) = 0, \quad S(\hat{\partial}_\rho) = -\hat{\partial}_\rho. \quad (14)$$

## ★-product approach

Idea: express everything in terms of commutative variables, but keep track of deformation

Using Poincaré-Birkhoff-Witt (PBW) theorem

$$\hat{f}(\hat{x}) \mapsto f(x) \quad (15)$$

and

$$\hat{f}(\hat{x})\hat{g}(\hat{x}) = \hat{f} \cdot \hat{g}(\hat{x}) \mapsto f \star g(x), \quad (16)$$

where

$$f \star g(x) = e^{\frac{i}{2} \frac{\partial}{\partial x^\rho} \theta^{\rho\sigma} \frac{\partial}{\partial y^\sigma}} f(x)g(y) \Big|_{y \rightarrow x}, \quad (17)$$

or

$$f \star g(x) = \sum_{n=1}^{\infty} \left(\frac{i}{2}\right)^n \frac{1}{n!} \theta^{\rho_1 \sigma_1} \dots \theta^{\rho_n \sigma_n} \left( \partial_{\rho_1} \dots \partial_{\rho_n} f(x) \right) \left( \partial_{\sigma_1} \dots \partial_{\sigma_n} g(x) \right) \quad (18)$$

is Moyal-Weyl ★-product  
 -associative but noncommutative  
 -under complex conjugation

$$\overline{f \star g} = \bar{g} \star \bar{f}. \quad (19)$$

Special example  $[\hat{x}^\mu, \hat{x}^\nu] \mapsto [x^\mu \star, x^\nu] = i\theta^{\mu\nu}$ .

## ★-product representation of derivatives

$$\begin{aligned}\hat{f} \cdot \hat{g}(\hat{x}) &\mapsto f \star g(x) \\ \hat{\partial}_\mu \hat{f}(\hat{x}) &\mapsto \underbrace{\partial_\mu^\star}_{?} f(x)\end{aligned}$$

How to do it:

$$\begin{array}{ccc} \hat{f}(\hat{x}) & \longmapsto & f(x) \\ \hat{\partial} \downarrow & & \downarrow \partial^\star \\ (\hat{\partial} \hat{f})(\hat{x}) & \longmapsto & (\partial^\star f)(x) \end{array} \quad (20)$$

For derivatives defined by (12)

$$(\partial_\rho^\star f)(x) = \partial_\rho f(x) \quad (21)$$

and

$$\partial_\rho^\star (f \star g(x)) = (\partial_\rho^\star f(x)) \star g(x) + f(x) \star (\partial_\rho^\star g(x)). \quad (22)$$

With (18), (21) and (22) one has enough ingredients to be able to start doing some physics.

## Deformed Lorentz transformations

classical Lorentz transformations  $x^\mu \rightarrow x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu$ ,  
 $\omega^{\mu\nu} = -\omega^{\nu\mu} = \text{const.}$

scalar field  $\phi(x) \rightarrow \phi'(x') = \phi(x)$ ,

$$\delta_\omega^{cl} \phi(x) \stackrel{\text{def}}{=} \phi'(x) - \phi(x) = -\omega^\lambda{}_\nu x^\nu \partial_\lambda \phi(x). \quad (23)$$

### Algebra of transformations

$$[\delta_\omega^{cl}, \delta_{\omega'}^{cl}] = \delta_{[\omega, \omega']}^{cl}, \quad (24)$$

### Leibniz rule

$$\begin{aligned} \delta_\omega^{cl} \left( \phi_1(x) \phi_2(x) \right) &= \left( \delta_\omega^{cl} \phi_1(x) \right) \phi_2(x) + \phi_1(x) \left( \delta_\omega^{cl} \phi_2(x) \right) \\ &= -\omega^\lambda{}_\nu x^\nu \partial_\lambda \left( \phi_1(x) \phi_2(x) \right). \end{aligned} \quad (25)$$

Idea: deform transformations (23) in such a way that can be easily rewritten in terms of elements form  $\hat{\mathcal{A}}_{\hat{x}}$ , that is, lifted to the space of noncommutative coordinates.

Rewrite  $\omega^\lambda{}_\nu x^\nu \partial_\lambda \phi(x)$  in terms of  $\star$ -product and  $\star$ -derivatives

$$\omega^\lambda{}_\nu x^\nu \partial_\lambda \phi = (X_\omega^\star \star \phi). \quad (26)$$

Solving (26) perturbatively one finds to all orders in  $\theta$

$$X_\omega^\star = (\omega^\lambda{}_\nu x^\nu) \partial_\lambda^\star - \frac{i}{2} \theta^{\rho\sigma} \omega^\lambda{}_\rho \partial_\sigma^\star \partial_\lambda^\star. \quad (27)$$

Deformed Lorentz transformation of a scalar field is

$$\begin{aligned} \delta_\omega \phi &= -X_\omega^\star \star \phi \\ &= -(\omega^\lambda{}_\nu x^\nu) \star (\partial_\lambda^\star \phi) + \frac{i}{2} \theta^{\rho\sigma} \omega^\lambda{}_\rho (\partial_\sigma^\star \partial_\lambda^\star \phi). \end{aligned} \quad (28)$$

Algebra of deformed transformations is undeformed

$$[\hat{\delta}_\omega, \hat{\delta}'_\omega] = \hat{\delta}_{[\omega, \omega']}. \quad (29)$$

However, demanding

$$\delta_\omega \left( \phi_1 \star \phi_2 \right) \stackrel{\text{def}}{=} -\omega^\lambda{}_\nu x^\nu \partial_\lambda \left( \phi_1 \star \phi_2 \right), \quad (30)$$

we get the deformed Leibniz rule

$$\begin{aligned} \delta_\omega \left( \phi_1 \star \phi_2 \right) &= (\delta_\omega \phi_1) \star \phi_2 + \phi_1 \star (\delta_\omega \phi_2) \\ &\quad - \frac{i}{2} \theta^{\rho\sigma} \left( (\delta_{\partial_\rho \omega} \phi_1) (\partial_\sigma^\star \phi_2) + (\partial_\rho^\star \phi_1) (\delta_{\partial_\sigma \omega} \phi_2) \right) \end{aligned} \quad (31)$$

where  $\delta_{\partial_\rho \omega} \phi_1 = -\omega^\lambda{}_\rho (\partial_\lambda \phi_1)$ .

## The abstract algebra

$$\hat{\delta}_\omega = -\omega^\lambda{}_\nu \hat{x}^\nu \hat{\partial}_\lambda + \frac{i}{2} \theta^{\rho\sigma} \omega^\lambda{}_\rho \hat{\partial}_\lambda \hat{\partial}_\sigma \quad (32)$$

consistent with (2) relations

$$\begin{aligned} \hat{\delta}_\omega \left( [\hat{x}^\mu, \hat{x}^\nu] - i\theta^{\mu\nu} \right) &= \left( [\hat{x}^\mu, \hat{x}^\nu] - i\theta^{\mu\nu} \right) \hat{\delta}_\omega \\ &+ \left( [\hat{x}^\alpha, \hat{x}^\nu] - i\theta^{\alpha\nu} \right) \omega^\mu{}_\alpha - \left( [\hat{x}^\alpha, \hat{x}^\mu] - i\theta^{\alpha\mu} \right) \omega^\nu{}_\alpha. \end{aligned} \quad (33)$$

Transformations close (undeformed) algebra

$$[\hat{\delta}_\omega, \hat{\delta}'_{\omega'}] = \hat{\delta}_{[\omega, \omega']}, \quad (34)$$

comultiplication is deformed

$$\begin{aligned} \Delta \hat{\delta}_\omega &= \hat{\delta}_\omega \otimes 1 + 1 \otimes \hat{\delta}_\omega \\ &+ \frac{i}{2} \theta^{\rho\sigma} \left( \omega^\lambda{}_\rho \hat{\partial}_\lambda \otimes \hat{\partial}_\sigma + \hat{\partial}_\rho \otimes \omega^\lambda{}_\sigma \hat{\partial}_\lambda \right). \end{aligned} \quad (35)$$

Result was also found by Chaichian et al., hep-th/0408069 and by Koch et al., hep-th/0409012.

From (35) one sees that deformed Lorentz transformations do not close bialgebra themselves but derivatives have to be included as well.

## Transformation law of a derivative

$$[\hat{\delta}_\omega, \hat{\partial}_\rho] = \omega^\lambda{}_\rho \hat{\partial}_\lambda. \quad (36)$$

## $\theta$ -deformed Poincaré bialgebra

$$[\hat{\partial}_\mu, \hat{\partial}_\nu] = 0, \quad [\hat{\delta}_\omega, \hat{\partial}_\rho] = \omega_\rho{}^\mu \hat{\partial}_\mu,$$

$$[\hat{\delta}_\omega, \hat{\delta}'_{\omega'}] = \hat{\delta}_{[\omega, \omega']},$$

$$\Delta \hat{\partial}_\rho = \hat{\partial}_\rho \otimes 1 + 1 \otimes \hat{\partial}_\rho,$$

$$\begin{aligned} \Delta \hat{\delta}_\omega &= \hat{\delta}_\omega \otimes 1 + 1 \otimes \hat{\delta}_\omega \\ &\quad + \frac{i}{2} \theta^{\rho\sigma} \left( \omega^\lambda{}_\rho \hat{\partial}_\lambda \otimes \hat{\partial}_\sigma + \hat{\partial}_\rho \otimes \omega^\lambda{}_\sigma \hat{\partial}_\lambda \right). \end{aligned}$$

## Fields

### Scalar field

$$\begin{aligned}\delta_\omega \phi &= -(X_\omega \star \phi) \\ &= -(\omega^\lambda{}_\nu x^\nu) \star (\partial_\lambda \phi) + \frac{i}{2} \theta^{\rho\sigma} \omega^\lambda{}_\rho \partial_\sigma \partial_\lambda \phi\end{aligned}\tag{37}$$

### Covariant vector field

$$\begin{aligned}\delta_\omega V_\mu &= -(X_\omega^\star \star V_\mu) - (X_{(\omega^\lambda{}_\mu)}^\star \star V_\lambda) \\ &= -(\omega^\lambda{}_\nu x^\nu) \star (\partial_\lambda V_\mu) + \frac{i}{2} \theta^{\rho\sigma} \omega^\lambda{}_\rho (\partial_\sigma \partial_\lambda V_\mu) - \omega^\lambda{}_\mu V_\lambda.\end{aligned}\tag{38}$$

### Contravariant vector field

$$\begin{aligned}\delta_\omega V^\mu &= -(X_\omega^\star \star V^\mu) + (X_{(\omega^\mu{}_\lambda)}^\star \star V^\lambda) \\ &= -(\omega^\lambda{}_\nu x^\nu) \star (\partial_\lambda V^\mu) + \frac{i}{2} \theta^{\rho\sigma} \omega^\lambda{}_\rho (\partial_\sigma \partial_\lambda V^\mu) + \omega^\mu{}_\lambda V^\lambda.\end{aligned}\tag{39}$$

Tensor fields analogously. . .

$$\begin{aligned}\delta_\omega(\phi_1 \star \phi_2) &= -(X_\omega \star (\phi_1 \star \phi_2)), \\ \delta_\omega(V_\mu \star \phi) &= -(X_\omega \star (V_\mu \star \phi)) - (X_{(\omega^\lambda{}_\mu)}^\star \star (V_\lambda \star \phi)) \\ &\quad \dots\end{aligned}$$

because of (31).

## Application

or an example of the action that is invariant under the deformed Lorentz transformations

Lagrangian for  $\phi^3$  theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu^* \phi) \star (\partial^{*\mu} \phi) - \frac{m^2}{2} \phi \star \phi - \lambda \phi \star \phi \star \phi \quad (40)$$

transforms covariantly because of (31)

$$\delta_\omega \mathcal{L} = -(X_\omega^* \star \mathcal{L}) = -\omega^\lambda{}_\nu x^\nu (\partial_\lambda \mathcal{L}). \quad (41)$$

One also needs an integral with the cyclic property

$$\int d^4x \phi_1 \star \phi_2 = \int d^4x \phi_2 \star \phi_1 = \int d^4x \phi_1 \phi_2. \quad (42)$$

The action

$$\begin{aligned} S &= \int d^4x \mathcal{L} \\ &= \int d^4x \left( \frac{1}{2}(\partial_\mu^* \phi) \star (\partial^{*\mu} \phi) - \frac{m^2}{2} \phi \star \phi - \lambda \phi \star \phi \star \phi \right) \end{aligned} \quad (43)$$

is invariant under deformed Lorentz transformations.

## Variational principle

$$\begin{aligned} \frac{\delta}{\delta g(x)} \int d^4x f \star g \star h &= \frac{\delta}{\delta g(x)} \int d^4x g(h \star f) \\ &= h \star f \end{aligned} \tag{44}$$

gives the equation of motion

$$(\partial^{\star\mu} \partial_{\mu}^{\star} \phi) + m^2 \phi + 3\lambda \phi \star \phi = 0. \tag{45}$$

Expanding the action ( $\star$ -products) and then varying with the respect to the field  $\phi$  gives the same result.

Conservation law (Nöther's theorem) not clear at the moment; energy-momentum tensor exists, it is conserved but not symmetric; work in progress.

## Summary and outlook

- it is possible to construct deformed Lorentz symmetry for the  $\theta$ -deformed space using the method of inverting the  $\star$ -product;  
deformed symmetry is deformation of classical Lorentz symmetry;  
algebra is undeformed, but comultiplication (Leibniz rules) changes
- tensor calculus established
- construction of invariant actions, analysis of conserved quantities in progress
- using the same method one deforms the classical diffeomorphism algebra and on that basis constructs a deformed theory of gravity, talk of Frank Meyer